Integral s parametrom

Naj bo $f(x,t)$: [a,b]×[c,d] → ℝ funkcija:

Zveznost

$f(x,t)$ zvezna na $D_f \implies$ F(x) = $\int_d^c f(x,t)dt$ zvezna na [a,b].

Odvedljivost

$f(x,t)$ zvezna in zvezno parcialno odvedljiva po x na $D_f \implies$ F(x) = $\int_d^c f(x,t)dt$ odvedljiva na [a,b] in velja:

$$F'(x) = \int_d^c f_x(x,t)dt.$$

Integrabilnost

$f(x,t)$ zvezna na $D_f \implies$ F(x) = $\int_c^d f(x,t)dt$ integrabilna na [a,b] in velja:

$$\int_b^a F(x)dx = \int_a^b (\int_d^c f(x,t)dt)dx = \int_c^d (\int_a^b f(x,t)dx)dt.$$

Integral z variabilnimi mejami

Naj bo $f(x,t)$: [a,b]×[c,d] → ℝ funkcija:

Odvedljivost

Naj bosta u,v: [a,b] → [c,d] odvedljivi $\implies$ $F(x) = \int_{u(x)}^{v(x)} f(x,t)dt$ odvedljiva in velja:

$$F'(x) = \int_{u(x)}^{v(x)} f_x(x,t)dt + v'(x)f(x,v(x)) - u'(x)f(x,u(x)).$$

Izlimitirani integral s parametrom

Integral s parametrom $F(x) = \int_a^\infty f(x,t)dt$ je **enakomerno konvergenten** za x ∈ [c,d], če za $\forall \varepsilon > 0$ $\exists b > a$, da velja:

$$\left|\int_b^\infty f(x,t)dt\right| < \varepsilon \quad \forall x \in [c,d].$$

Weierstrass M-test

Če $\exists g$: [a,∞) → ℝ, da velja $|f(x,t)| < g(t)$ za $\forall x \in$ [c,d] in je $\int_a^\infty g(t)dt < \infty \implies F(x) = \int_a^\infty f(x,t)dt$ enakomerno konvergenten na [c,d]. V pomoč je formula:

$$\int_0^* t^{-\alpha}dt < \infty \iff \alpha < 1.$$

Naj bo $f(x,t)$: [c,d]×[a,∞) → ℝ funkcija:

Zveznost

$f(x,t)$ zvezna na D_f in $F(x) = \int_a^\infty f(x,t)dt$ enakomerno konvergentna na [c,d] $\implies F$ zvezna na [c,d].

Odvedljivost

$f(x,t)$ zvezna in zvezno parcialno odvedljiva po x na D_f, $F(x) = \int_a^\infty f(x,t)dt$ konvergentna na [c,d] ter $F(x) = \int_a^\infty f_x(x,t)dt$ enakomerno konvergentna na [c,d] $\implies F'(x) = \int_a^\infty f_x(x,t)dt$.

Integrabilnost

$f(x,t)$ zvezna na D_f in $F(x) = \int_a^\infty f_x(x,t)dt$ enakomerno konvergentna na [c,d], potem velja:

$$\int_c^d F(x)dx = \int_c^d (\int_a^\infty f(x,t)dt)dx = \int_a^\infty (\int_c^d f(x,t)dx)dt.$$

Gamma in Beta funkciji

Gamma funkcija:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \quad x \in (0,\infty)$$

1. $\Gamma(x+1) = x\Gamma(x)\ \forall x > 0.$
2. $\Gamma(n) = (n-1)!\ \forall n \in \mathbb{N}.$
3. $\Gamma(\frac{1}{2}) = \sqrt{\pi}.$

Beta funkcija:

$$\beta(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt \quad p,q > 0$$

1. $\beta(p,q) = \frac{\Gamma(p)\cdot\Gamma(q)}{\Gamma(p+q)}.\ \forall p,q > 0.$
2. $\beta(p,q) = \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}}du.$
3. $\int_0^{\pi/2} sin^{2p-1}x \cdot cos^{2q-1}xdx = \frac{1}{2}\beta(p,q) \quad p,q > 0.$
4. $\beta(1,q) = \frac{1}{q}.$
5. $\beta(p+1,q) = \frac{p}{p+q}\beta(p,q).$
6. $\beta(p,q) = \beta(q,p).$
Eulerjeva refleksijska formula: $\beta(p,1-p) = \frac{\pi}{sin(\pi p)}$ $p \in (0,1)$.

Integrali

$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1}(n \neq -1)$
$\frac{1}{x}$	$\ln|x|$
a^x	$a^x \ln a$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\sec x \tan x$	$\frac{1}{\sec x}$
$\frac{1}{\cos^2 x}$	$\tan x$
$\frac{1}{\sin^2 x}$	$-\cot x$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
$\frac{1}{\sqrt{1+x^2}}$	$\sinh^{-1} x$
$\frac{1}{1+x^2}$	$\arctan x$

Per Partes

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$
$$\int udv = uv - \int vdu$$

Racionalne funkcije

$\int \frac{p(x)}{q(x)}dx$, $p(x)$, $q(x)$ sta polinoma

1. Če je $st(q(x)) <= st(p(x))$ polinoma delimo
2. $q(x)$ razdelimo na linearne in kvadratne faktorje
3. Izraz pod integralom razcepimo na parcialne ulomke $\frac{p(x)}{q(x)} = \left[\frac{A_1}{x-a_1} + \cdots + \frac{A_{n_1}}{(x-a_1)^{n_1}}\right]$
4. Integriramo vsakega zase

$$k \geq 2 \qquad st(p(x)) \leq 2k-1$$
$$st(q(x)) \leq 2k-3 \qquad (ax^2+bx+c) \qquad \text{nerazcepen v } \mathbb{R}$$
$$I = \int \frac{p(x)}{(ax^2+bx+c)^k} = \int \frac{Ax+B}{ax^2+bx+c} + \frac{q(x)}{(ax^2+bx+c)^{k-1}}$$

A,B, $q(x)$ poiščemo tako da enačbo odvajamo.

Korenske funkcije

1. $\int f(\sqrt{ax+b})dx \qquad t = \sqrt{ax+b}$
2. $\int f(\sqrt{ax^2+bx+c})dx$
   (a) $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ ga prevedemo na oblike:
   - Če je $a < 0 : \int \frac{dx}{\sqrt{1-x^2}} = \arcsin x$
   - Če je $a > 0 : \int \frac{dx}{\sqrt{x^2+c}} = \ln\left|x + \sqrt{x^2+c}\right|$

   (b) $\int \frac{p(x)}{\sqrt{ax^2+bx+c}} = q(x)\sqrt{ax^2+bx+c} + A\int \frac{dx}{\sqrt{ax^2+bx+c}}$
   $st(p(x)) - 1 = st(q(x))$ A, q(x) poiščemo z odvanjanjem
3. $\int \sqrt{a^2-x^2}dx \quad x = a\sin t \quad dx = a\cos tdt \quad t = \arcsin\frac{x}{a}$
4. $\int \sqrt{a^2+x^2}dx \quad x = a\,\mathrm{sh}\,t \quad dx = a\,\mathrm{ch}\,tdt \quad t = \mathrm{arsh}\frac{x}{a}$

Kotne funkcije

$$\int \sin(ax)\sin(bx)dx = \int -\frac{1}{2}\left[\cos(a+b)x - \cos(a-b)x\right]dx =$$
$$= -\frac{1}{2}\left[\frac{sin(a-b)x}{(a-b)} - \frac{sin(a+b)x}{(a+b)}\right]$$
$$\int \cos(ax)\cos(bx)dx = \int \frac{1}{2}\left[\cos(a+b)x + \cos(a-b)x\right]dx \dots$$
$$\int \sin(ax)\cos(bx)dx = \int \frac{1}{2}\left[\sin(a+b)x + \sin(a-b)x\right]dx \dots$$

Lihe in sode kotne funkcije

$\int \cos^m x \sin^n x dx$

1. eno od števil m,n je liho (npr. $m = 2k+1$)

$$\int \cos^{2k} x \cos x \sin^n x dx = \int t^n (1-t^2)^k dt$$
$$t = \sin x \quad dt = \cos x dx$$
$$cos^{2k}x = (cos^2 x)^k = (1-t^2)^k$$

2. m, n sta oba soda, $m = 2m_1$, $n = 2n_1$

$$\int \cos^{2m_1} x \sin^{2n_1} x dx = \int (\cos^2 x)^{m_1} (\sin^2 x)^{n_1} dx =$$
$$= \int \left(\frac{1+\cos 2x}{2}\right)^{m_1} \left(\frac{1-\cos 2x}{2}\right)^{n_1} =$$
$$= \text{vsota integralov oblike} \int \cos^k 2x dx$$

kjer je $k \le m_1 + n_1 = \frac{1}{2}(m+n) < m+1$
če je k lih gremo po 1 točki
če je k sod ponovimo postopek

3. $\int R(\cos x, \sin x)dx$ ($R\dots$ racinonalni izraz)

$$t = \tan\frac{x}{2} \quad \cos x = \frac{1-t^2}{t^2+1}$$
$$\sin x = \frac{2t}{t^2+1} \quad dx = \frac{2}{t^2+1}dt$$
$$t = \tan x \quad \cos x = \frac{1}{\sqrt{t^2+1}}$$
$$\sin x = \frac{t}{\sqrt{t^2+1}} \quad dx = \frac{dt}{t^2+1}$$

Znane limite

$\lim_{x\to\infty} a^x = 0$, $\lvert a\rvert < 1$	$\lim_{x\to 0} x^x = 1$	$\lim_{x\to\infty} \sqrt[x]{x} = 1$
$\lim_{x\to 0} \frac{a^x-1}{x} = \ln a$, $a > 0$	$\lim_{x\to\infty} \frac{\ln x}{x} = 0$	$\lim_{x\to 0} x \ln x = 0$
$\lim_{x\to\infty} \left(1+\frac{k}{x}\right)^{mx} = e^{mk}$	$\lim_{x\to 0} \left(1+kx\right)^{\frac{m}{x}} = e^{mk}$	$\lim_{x\to 0} \frac{\sin x}{x} = 1$

Prevladujoči členi

$n^n \gg n! \gg q^n$ ($\lvert q\rvert > 1$) $\gg n^a$ ($a > 0$) $\gg ln(n)^a$ ($a > 0$)

Znani odvodi

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
x^n	nx^{n-1}	a^x	$a^x \ln(a)$	e^x	e^x
$\frac{1}{x^n}$	$-\frac{n}{x^{n-1}}$	$\ln x$	$\frac{1}{x}$	$\log_a x$	$\frac{1}{xlna}$
$\sin x$	$\cos x$	$\cos x$	$-\sin x$	$\tan x$	$\frac{1}{\cos^2 x}$
$\sec x$	$\tan(x)\sec(x)$	$\csc x$	$-\cot(x)\csc(x)$	$\cot x$	$-\frac{1}{\sin^2 x}$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$	$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$	$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\sinh x$	$\cosh x$	$\cosh x$	$\sinh x$	$\tanh x$	$\frac{1}{\cosh^2 x}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$	$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$	$\tanh^{-1} x$	$\frac{1}{1-x^2}$

Znane vrste

$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$	$\mathbb{R}$	$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$	$\mathbb{R}$
$\sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$	$\mathbb{R}$	$ln(1-x) = -\sum_{n=1}^{\infty} \frac{1}{n} x^n$	$(-1{,}1)$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$\mathbb{R}$	$ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$	$(-1{,}1)$
$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$	$(-1{,}1)$	$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$	$(-1{,}1)$
$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n$	$(-1{,}1)$	$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$	$(-1{,}1)$

Funkcije

Krožne funkcije

$\sin^{-1} x$	$D_f = [-1{,}1]$	$Z_f = [-\frac{\pi}{2}, \frac{\pi}{2}]$
$\cos^{-1} x$	$D_f = [-1{,}1]$	$Z_f = [0{,}\pi]$
$\tan^{-1} x$	$D_f = (-\infty{,}\infty)$	$Z_f = (-\frac{\pi}{2}, \frac{\pi}{2})$
$\cot^{-1} x$	$D_f = (-\infty{,}\infty)$	$Z_f = (0{,}\pi)$
$\sec^{-1} x$	$D_f = (-\infty, -1] \cup [1{,}\infty)$	$Z_f = [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$
$\csc^{-1} x$	$D_f = (-\infty, -1] \cup [1{,}\infty)$	$Z_f = [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$

Hiperbolične funkcije

$\sinh x$	$\frac{e^x - e^{-x}}{2} = \frac{e^{2x}-1}{2e^x} = \frac{1-e^{-2x}}{2e^{-x}}$	
$\cosh x$	$\frac{e^x + e^{-x}}{2} = \frac{e^{2x}+1}{2e^x} = \frac{1+e^{-2x}}{2e^{-x}}$	
$\sinh^{-1} x$	$\ln(x + \sqrt{x^2+1})$	
$\cosh^{-1} x$	$\ln(x + \sqrt{x^2-1})$	$x \ge 1$
$\tanh^{-1} x$	$\frac{1}{2}\ln(\frac{1+x}{1-x})$	$\lvert x\rvert < 1$
$\coth^{-1} x$	$\frac{1}{2}\ln(\frac{x+1}{x-1})$	$\lvert x\rvert > 1$
$\cosh x + \sinh x = e^x$	$\cosh^2 x - \sinh^2 x = 1$	
$\cosh x - \sinh x = e^{-x}$		

*identitete kotnih funkcij, vendar se pri sinh(x) * sinh(y) obrne predznak*

Kotne funkcije

	$0°$	$30°$	$45°$	$60°$	$90°$					
	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	Q1	Q2	Q3	Q4	S/L
$\sin\alpha$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	+	+	-	-	L
$\cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	+	-	-	+	S
$\tan\alpha$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$-$	+	-	+	-	L
$\cot\alpha$	$-$	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	+	-	+	-	L

$\sin\alpha = \frac{N}{H}$ $\quad \cos\alpha = \frac{P}{H}$ $\quad tan\alpha = \frac{N}{P}$ $\quad \cot\alpha = \frac{P}{N}$

$\sin^2\alpha = 1 - \cos^2\alpha$ $\quad \cos^2\alpha = 1 - \sin^2\alpha$

$\frac{1}{\sin^2\alpha} = 1 + \cot^2\alpha$ $\quad \frac{1}{\cos^2\alpha} = 1 + \tan^2\alpha$

$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1}{2}(1-\cos\alpha)}$ $\quad \tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}$

$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1}{2}(1+\cos\alpha)}$ $\quad \cot\frac{\alpha}{2} = \frac{\sin\alpha}{1-\cos\alpha}$

$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$ $\quad \cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$

$\cos(\alpha \mp \beta) = \cos\alpha\cos\beta \pm \sin\alpha\sin\beta$

$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

$\sin\alpha \pm \sin\beta = 2\sin\frac{\alpha\pm\beta}{2}\cos\frac{\alpha\mp\beta}{2}$

$\cos\alpha \pm \cos\beta = 2\cos\frac{\alpha\pm\beta}{2}\cos\frac{\alpha\mp\beta}{2}$

$\cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta) + \cos(\alpha-\beta)\right)$

$\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$

$\sin\alpha\sin\beta = -\frac{1}{2}\left(\cos(\alpha+\beta) - \cos(\alpha-\beta)\right)$

$\sin\alpha - \cos\beta = -2\sin(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$